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DIRECT COMPUTATIONAL APPROACHES FOR SOLVING SECOND-ORDER INITIAL VALUE PROBLEMS

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Abstract

This study presents the development and analysis of a new block hybrid method for the direct solution of second-order initial value problems (IVPs) of ordinary differential equations (ODEs). The method is constructed to improve accuracy and stability in solving highly stiff and oscillatory systems that frequently arise in science and engineering applications. The theoretical analysis establishes the consistency, zero-stability, and convergence of the proposed scheme. To demonstrate its efficiency, the method is applied to a set of real-life problems and selected highly stiff test equations of the second order. Numerical results are compared with existing methods. The findings reveal that the new method provides smaller error magnitudes and better error control than its counterparts, particularly in stiff regimes where conventional techniques often fail. This confirms the robustness and reliability of the proposed method for solving second-order IVPs.

Keywords: Block hybrid method; second-order ODEs; initial value problems; stiff systems; numerical analysis; stability; convergence.

1. INTRODUCTION

Mathematical modeling is now widely used to explain processes in biology, physics, and medicine. It has shown great value in studying areas like dynamical systems, body temperature regulation, and simple harmonic motion. As highlighted by Sabo et al. (2021a) and Elazzouzi et al. (2019), these models have advanced knowledge in both mathematics and the biosciences. The use of mathematical tools in such fields has also encouraged new ways of thinking and stronger collaboration across disciplines. In particular, studies of dynamic and thermal processes have benefited greatly, as mathematics helps provide clearer insights and more accurate predictions (Kubuye& Omar, 2015a; Sabo et al. 2021b).

This research is concerned with addressing initial value problems (IVPs) involving second-order ordinary differential equations (ODEs), expressed in the general form:

$$y''(t) = f(t, y, y'), y(t_0) = y_0, y'(t_0) = y_0', \;\; t \in [t_0, t_n] \tag{1.1}$$

Where t_0 represents initial value/point, y_0 denotes the solution at time t_0, f remains continuous over the integration interval. We operate under the assumption that equation (1.1) adheres to the existence and uniqueness theorem of differential equations. Furthermore, we presume that solutions to equations akin to (1.1) remain bounded. It's crucial to clarify that a solution $y(t)$ to equation (1.1) is deemed bounded if,

$$\sup_{t \in \Re} \| y(t) \| < \infty \tag{1.2}$$

The numerical approximation of equations such as (1.1) has attracted considerable attention from scholars, as discussed in the works of Kyagya et al. (2021), Kubuye and Omar (2015b), Adeyeye and Omar (2017), and Skwame et al. (2019).

Studies have demonstrated that solving an equation such as (1.1) in its original form is often more effective than transforming it into a system of first-order ordinary differential equations, as emphasized by Kwari et al. (2023). This understanding has motivated many researchers to develop direct methods that address equation (1.1) without the need for such reductions. A wide range of approaches for directly solving equation (1.1) have been reported in the literature. Noteworthy contributions include those by Adeniran and Ogundare (2015), Omole and Ogunware (2018), Kwanamu et al. (2021), Donald et al. (2021), Ayinde et al. (2023), and Ishaq et al. (2024). Additional significant works include Aloko et al. (2024), Adewale and Sabo (2024), as well as Sabo et al. (2024). Further efforts have been presented by Raymond et al. (2018), Ibrahim (2017), and Kamo et al. (2018). Collectively, these studies highlight the growing recognition and progress in the development of direct techniques for efficiently solving second-order ordinary differential equations.

2 COMPUTATIONAL BLOCK MATHEMATICAL FORMULATIONS

An approximate solution to a power series polynomial of the for

$$y(t) = \sum_{j=0}^{u+v-1} \alpha_j t^j \tag{2.1}$$

is considered as a basis function for the direct solution of the second initial value problems of the form (1.1). Where $t \in [a,b]$, the $a's$ are real unknown parameters to be determined and $u+v$ is the sum of the number of interpolation and collocation points. differentiating (2.1) twice, yield

$$y''(t) = \sum_{j=0}^{u+v-1} j(j-1)a_j t^{j-2} \tag{2.2}$$

Now, interpolating (2.1) at point $u = \frac{1}{2}, 1$ and collocating (2.2) at $v = 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1$ which lead to a system of equation in a matrix form as

$$\begin{bmatrix}
1 & t_{n+\frac{1}{2}} & t^2_{n+\frac{1}{2}} & t^3_{n+\frac{1}{2}} & t^4_{n+\frac{1}{2}} & t^5_{n+\frac{1}{2}} & t^6_{n+\frac{1}{2}} & t^7_{n+\frac{1}{2}} & t^8_{n+\frac{1}{2}} & t^9_{n+\frac{1}{2}} \\
1 & t_{n+1} & t^2_{n+1} & t^3_{n+1} & t^4_{n+1} & t^5_{n+1} & t^6_{n+1} & t^7_{n+\frac{1}{2}} & t^8_{n+1} & t^9_{n+1} \\
0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_n^5 & 56t_n^6 & 72t_n^7 \\
0 & 0 & 2 & 6t_{n+\frac{1}{6}} & 12t^2_{n+\frac{1}{6}} & 20t^3_{n+\frac{1}{6}} & 30t^4_{n+\frac{1}{6}} & 42t^5_{n+\frac{1}{6}} & 56t^6_{n+\frac{1}{6}} & 72t^7_{n+\frac{1}{6}} \\
0 & 0 & 2 & 6t_{n+\frac{1}{4}} & 12t^2_{n+\frac{1}{4}} & 20t^3_{n+\frac{1}{4}} & 30t^4_{n+\frac{1}{4}} & 42t^5_{n+\frac{1}{4}} & 56t^6_{n+\frac{1}{4}} & 72t^7_{n+\frac{1}{4}} \\
0 & 0 & 2 & 6t_{n+\frac{1}{3}} & 12t^2_{n+\frac{1}{3}} & 20t^3_{n+\frac{1}{3}} & 30t^4_{n+\frac{1}{3}} & 42t^5_{n+\frac{1}{3}} & 56t^6_{n+\frac{1}{3}} & 72t^7_{n+\frac{1}{3}} \\
0 & 0 & 2 & 6t_{n+\frac{1}{2}} & 12t^2_{n+\frac{1}{2}} & 20t^3_{n+\frac{1}{2}} & 30t^4_{n+\frac{1}{2}} & 42t^5_{n+\frac{1}{2}} & 56t^6_{n+\frac{1}{2}} & 72t^7_{n+\frac{1}{2}} \\
0 & 0 & 2 & 6t_{n+\frac{2}{3}} & 12t^2_{n+\frac{2}{3}} & 20t^3_{n+\frac{2}{3}} & 30t^4_{n+\frac{2}{3}} & 42t^5_{n+\frac{2}{3}} & 56t^6_{n+\frac{2}{3}} & 72t^7_{n+\frac{2}{3}} \\
0 & 0 & 2 & 6t_{n+\frac{3}{4}} & 12t^2_{n+\frac{3}{4}} & 20t^3_{n+\frac{3}{4}} & 30t^4_{n+\frac{3}{4}} & 42t^5_{n+\frac{3}{4}} & 56t^6_{n+\frac{3}{4}} & 72t^7_{n+\frac{3}{4}} \\
0 & 0 & 2 & 6t_{n+1} & 12t^2_{n+1} & 20t^3_{n+1} & 30t^4_{n+1} & 42t^5_{n+1} & 56t^6_{n+1} & 72t^7_{n+1}
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix}
=
\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+2} \\ f_n \\ f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix} \tag{2.3}$$

using Gaussian elimination method, (2.3) is solved for the $a_j's$. The values of the $a_j's$ obtained are then substituted into (1.1), after some manipulations, this gives a continuous hybrid linear multistep method of the form;

$$y(t) = \alpha_{\frac{1}{2}}(t)y_{n+\frac{1}{2}} + \alpha_1(t)y_{n+1} + h^2\left[\sum_{j=0}^{1}\beta_j(t)f_{n+j} + \beta_{v_i}(t)f_{n+v_i}\right], v_i = 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \tag{2.4}$$

the coefficient $\alpha_{\frac{1}{2}}, \alpha_1, \beta_0, \beta_{\frac{1}{6}}, \beta_{\frac{1}{4}}, \beta_{\frac{1}{3}}, \beta_{\frac{1}{2}}, \beta_{\frac{2}{3}}, \beta_{\frac{3}{4}}, \beta_1$ are given by;

$$\alpha_{\frac{1}{2}} = 2 - 2t$$

$$\alpha_1 = 2t - 1$$

$$\beta_0 = \frac{41}{13440} - \frac{1867}{40320}t + \frac{1}{2}t^2 - \frac{113}{36}t^3 + \frac{427}{36}t^4 - \frac{3367}{120}t^5 + \frac{1876}{45}t^6 - \frac{113}{3}t^7 + \frac{132}{7}t^8 - 4t^9$$

$$\beta_{\frac{1}{6}} = -\frac{54}{175}t + \frac{648}{35}t^3 - \frac{594}{5}t^4 + \frac{9072}{25} - \frac{3132}{5}t^6 + \frac{21816}{35}t^7 - \frac{11664}{35}t^8 + \frac{2592}{35}t^9$$

$$\beta_{\frac{1}{4}} = \frac{2}{21} + \frac{26}{175}t - \frac{512}{15}t^3 + \frac{11392}{45}t^4 - \frac{21248}{25}t^5 + \frac{14080}{9}t^6 - \frac{171008}{105}t^7 + \frac{31488}{35}t^8 - \frac{1024}{5}t^9$$

$$\beta_{\frac{1}{3}} = -\frac{243}{4480} - \frac{567}{3200}t + \frac{243}{10}t^3 - \frac{1539}{8}t^4 + \frac{138267}{200}t^5 - \frac{6723}{5}t^6 + \frac{51111}{35}t^7 - \frac{5832}{7}t^8 + \frac{972}{5}t^9$$

$$\beta_{\frac{1}{2}} = \frac{17}{105} - \frac{9}{35}t - 8t^3 + \frac{202}{3}t^4 - \frac{1304}{5}t^5 + \frac{8252}{15}t^6 - \frac{4504}{7}t^7 + \frac{2736}{7}t^8 - 96t^9$$

$$\beta_{\frac{2}{3}} = -\frac{243}{4480} + \frac{1647}{22400}t + \frac{81}{20}t^3 - \frac{351}{10}t^4 + \frac{28269}{200}t^5 - \frac{1566}{5}t^6 + \frac{13581}{35}t^7 - \frac{8748}{35}t^8 + \frac{324}{5}t^9$$

$$\beta_{\frac{3}{4}} = \frac{2}{21} - \frac{278}{1575}t + \frac{512}{315}t^3 - \frac{128}{9}t^4 + \frac{4352}{75}t^5 - \frac{5888}{45}t^6 + \frac{17408}{105}t^7 - \frac{768}{7}t^8 - \frac{1024}{35}t^9$$

$$\beta_1 = \frac{41}{13440} - \frac{143}{22400}t + \frac{1}{30}t^3 - \frac{107}{360}t^4 + \frac{249}{200}t^5 - \frac{131}{45}t^6 + \frac{407}{405}t^7 - \frac{96}{35}t^8 + \frac{4}{5}t^9$$

evaluating (2.4) at non interpolating points to obtain the continuous form as,

$$\left.\begin{aligned}
&y_n-2y_{n+\frac{1}{2}}-y_{n+1}=\frac{41}{13440}h^2f_n+\frac{2}{21}h^2f_{n+\frac{1}{4}}-\frac{243}{4480}h^2f_{n+\frac{1}{3}}+\frac{17}{105}h^2f_{n+\frac{1}{2}}-\frac{243}{4480}h^2f_{n+\frac{2}{3}}+\frac{2}{21}f_{n+\frac{3}{4}}+\frac{41}{13440}f_{n+1}\\
&y_{n+\frac{1}{6}}-\frac{5}{3}y_{n+\frac{1}{2}}-+\frac{2}{3}y_{n+1}=\frac{8851}{8817984}h^2f_n-\frac{188}{8505}h^2f_{n+\frac{1}{6}}+\frac{52522}{688905}h^2f_{n+\frac{1}{4}}-\frac{4979}{90720}h^2f_{n+\frac{1}{3}}+\frac{10117}{91854}h^2f_{n+\frac{1}{2}}-\frac{20473}{544320}h^2f_{n+\frac{2}{3}}+\frac{44146}{688905}h^2f_{n+\frac{3}{4}}+\frac{22283}{11022480}h^2f_{n+1}\\
&y_{n+\frac{1}{4}}-\frac{3}{2}y_{n+\frac{1}{2}}+\frac{1}{2}y_{n+1}=\frac{1749}{2293760}h^2f_n-\frac{4941}{286720}h^2f_{n+\frac{1}{6}}+\frac{359}{6720}h^2f_{n+\frac{1}{4}}-\frac{106191}{2293760}h^2f_{n+\frac{1}{3}}+\frac{23393}{286720}h^2f_{n+\frac{1}{2}}-\frac{64233}{2293760}h^2f_{n+\frac{2}{3}}+\frac{43}{896}h^2f_{n+\frac{3}{4}}+\frac{10441}{6881280}h^2f_{n+1}\\
&y_{n+\frac{1}{3}}-\frac{4}{3}y_{n+\frac{1}{2}}+\frac{1}{3}y_{n+1}=\frac{22819}{44089920}h^2f_n-\frac{20}{1701}h^2f_{n+\frac{1}{6}}+\frac{25052}{688905}h^2f_{n+\frac{1}{4}}-\frac{6737}{181440}h^2f_{n+\frac{1}{3}}+\frac{12182}{229635}h^2f_{n+\frac{1}{2}}-\frac{10019}{544320}h^2f_{n+\frac{2}{3}}+\frac{628}{19683}h^2f_{n+\frac{3}{4}}+\frac{44659}{44089920}h^2f_{n+1}\\
&y_{n+\frac{2}{3}}-\frac{2}{3}y_{n+\frac{1}{2}}-\frac{1}{3}y_{n+1}=-\frac{45149}{88179840}h^2f_n+\frac{20}{1701}h^2f_{n+\frac{1}{6}}-\frac{5098}{137781}h^2f_{n+\frac{1}{4}}+\frac{2963}{72576}h^2f_{n+\frac{1}{3}}-\frac{7211}{229635}h^2f_{n+\frac{1}{2}}+\frac{24061}{1088640}h^2f_{n+\frac{2}{3}}-\frac{22418}{688905}h^2f_{n+\frac{3}{4}}-\frac{88829}{88179840}f_{n+1}\\
&y_{n+\frac{3}{4}}-\frac{1}{2}y_{n+\frac{1}{2}}-\frac{1}{2}y_{n+1}=-\frac{5179}{6881280}h^2f_n+\frac{4941}{286720}h^2f_{n+\frac{1}{6}}-\frac{727}{13440}h^2f_{n+\frac{1}{4}}+\frac{136323}{2293760}h^2f_{n+\frac{1}{3}}-\frac{37883}{860160}h^2f_{n+\frac{1}{2}}+\frac{18873}{458752}h^2f_{n+\frac{2}{3}}-\frac{109}{2240}h^2f_{n+\frac{3}{4}}-\frac{10373}{6881280}f_{n+1}
\end{aligned}\right\}\quad(2.5)$$

differentiating (2.4) once, yields

$$y'(t)=\sigma_{\frac{1}{2}}(t)y_{n+\frac{1}{2}}+\sigma_1(t)y_{n+1}+h^2\left[\sum_{j=0}^{1}\beta'_j(t)f_{n+j}+\beta'_{v_i}(t)f_{n+v_i}\right],v_i=0,\frac{1}{6},\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4}\quad(2.6)$$

The coefficient $\sigma_{\frac{1}{2}},\sigma_1,\beta'_0,\beta'_{\frac{1}{6}},\beta'_{\frac{1}{4}},\beta'_{\frac{1}{3}},\beta'_{\frac{1}{2}},\beta'_{\frac{2}{3}},\beta'_{\frac{3}{4}},\beta'_1$ are given by;

$$\begin{aligned}
\alpha'_{\frac{1}{2}}&=-2\\
\alpha'_1&=2\\
\beta'_0&=-\frac{1867}{40320}+t-\frac{113}{12}t^2+\frac{427}{9}t^3-\frac{3367}{24}t^4+\frac{3752}{15}t^5-\frac{791}{3}t^6+\frac{1056}{7}t^7-36t^8\\
\beta'_{\frac{1}{6}}&=-\frac{54}{175}+\frac{1944}{35}t^2-\frac{2376}{5}t^3+\frac{9072}{5}t^4-\frac{18792}{5}t^5+\frac{21816}{5}t^6-\frac{93312}{35}t^7+\frac{23328}{35}t^8\\
\beta'_{\frac{1}{4}}&=\frac{26}{175}-\frac{512}{5}t^2+\frac{45568}{45}t^3-\frac{21248}{5}t^4+\frac{28160}{3}t^5-\frac{171008}{15}t^6+\frac{251904}{35}t^7-\frac{9216}{5}t^8\\
\beta'_{\frac{1}{3}}&=-\frac{567}{3200}+\frac{729}{10}t^2-\frac{1539}{2}t^3+\frac{138267}{40}t^4-\frac{40338}{5}t^5+\frac{51111}{5}t^6-\frac{46656}{7}t^7+\frac{8748}{5}t^8\\
\beta'_{\frac{1}{2}}&=-\frac{143}{22400}+\frac{1}{10}t^2-\frac{107}{90}t^3+\frac{249}{40}t^4-\frac{262}{15}t^5+\frac{407}{15}t^6-\frac{768}{35}t^7+\frac{36}{5}t^8\\
\beta'_{\frac{2}{3}}&=-\frac{1647}{22400}+\frac{243}{20}t^2-\frac{702}{5}t^3+\frac{28269}{40}t^4-\frac{9396}{5}t^5+\frac{13581}{5}t^6-\frac{69984}{35}t^7+\frac{2916}{5}t^8\\
\beta'_{\frac{3}{4}}&=-\frac{278}{1575}-\frac{512}{105}t^2+\frac{512}{9}t^3-\frac{4352}{15}t^4+\frac{11776}{15}t^5-\frac{17408}{15}t^6+\frac{6144}{7}t^7-\frac{9216}{35}t^8\\
\beta'_1&=-\frac{143}{22400}+\frac{1}{10}t^2-\frac{107}{90}t^3+\frac{249}{40}t^4-\frac{626}{15}t^5+\frac{407}{15}t^6-\frac{768}{35}t^7+\frac{36}{5}t^8
\end{aligned}$$

On evaluating (2.6) at all point, so that the following discrete methods are obtained

$$\left.\begin{aligned}
&hy'_n+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{1867}{40320}hf_n-\frac{54}{175}hf_{n+\frac{1}{6}}+\frac{26}{175}hf_{n+\frac{1}{4}}-\frac{567}{3200}hf_{n+\frac{1}{3}}-\frac{9}{35}hf_{n+\frac{1}{2}}+\frac{1647}{22400}hf_{n+\frac{2}{3}}-\frac{278}{1575}hf_{n+\frac{3}{4}}-\frac{143}{22400}hf_{n+1}\\
&hy'_{n+\frac{1}{6}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{13609}{4898880}hf_n+\frac{167}{4725}hf_{n+\frac{1}{6}}-\frac{114598}{382725}hf_{n+\frac{1}{4}}+\frac{5611}{50400}hf_{n+\frac{1}{3}}-\frac{8789}{25515}hf_{n+\frac{1}{2}}-\frac{35107}{302400}hf_{n+\frac{2}{3}}-\frac{74014}{382725}hf_{n+\frac{3}{4}}-\frac{18511}{3061800}hf_{n+1}\\
&hy'_{n+\frac{1}{4}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{1711}{573440}hf_n+\frac{24273}{358400}hf_{n+\frac{1}{6}}-\frac{23659}{100800}hf_{n+\frac{1}{4}}+\frac{273213}{2867200}hf_{n+\frac{1}{3}}-\frac{73343}{215040}hf_{n+\frac{1}{2}}+\frac{46953}{409600}hf_{n+\frac{2}{3}}-\frac{6479}{33600}hf_{n+\frac{3}{4}}-\frac{156281}{25804800}hf_{n+1}\\
&hy'_{n+\frac{1}{3}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{811}{279936}hf_n+\frac{302}{4725}hf_{n+\frac{1}{6}}-\frac{72418}{382725}hf_{n+\frac{1}{4}}+\frac{28039}{201600}hf_{n+\frac{1}{3}}-\frac{8777}{25515}hf_{n+\frac{1}{2}}+\frac{69989}{604800}hf_{n+\frac{2}{3}}-\frac{73954}{382725}hf_{n+\frac{3}{4}}-\frac{296341}{48988800}hf_{n+1}\\
&hy'_{n+\frac{1}{2}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{67}{20160}hf_n+\frac{27}{350}hf_{n+\frac{1}{6}}-\frac{386}{1575}hf_{n+\frac{1}{4}}+\frac{1539}{5600}hf_{n+\frac{1}{3}}-\frac{11}{42}hf_{n+\frac{1}{2}}+\frac{1161}{11200}hf_{n+\frac{2}{3}}-\frac{298}{1575}hf_{n+\frac{3}{4}}-\frac{11}{1800}hf_{n+1}\\
&hy'_{n+\frac{2}{3}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{27569}{9797760}hf_n+\frac{302}{4725}hf_{n+\frac{1}{6}}-\frac{10894}{54675}hf_{n+\frac{1}{4}}+\frac{436339}{201600}hf_{n+\frac{1}{3}}-\frac{3713}{25515}hf_{n+\frac{1}{2}}+\frac{116789}{604800}hf_{n+\frac{2}{3}}-\frac{77794}{382725}hf_{n+\frac{3}{4}}-\frac{292261}{48988800}hf_{n+1}\\
&hy'_{n+\frac{3}{4}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{3067}{1032192}hf_n+\frac{24273}{358400}hf_{n+\frac{1}{6}}-\frac{2371}{11200}hf_{n+\frac{1}{4}}+\frac{662013}{2867200}hf_{n+\frac{1}{3}}-\frac{11349}{71680}hf_{n+\frac{1}{2}}+\frac{717471}{2867200}hf_{n+\frac{2}{3}}-\frac{17117}{100800}hf_{n+\frac{3}{4}}-\frac{17329}{2867200}hf_{n+1}\\
&hy'_{n+1}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{61}{4480}hf_n-\frac{54}{175}hf_{n+\frac{1}{6}}+\frac{1514}{1575}hf_{n+\frac{1}{4}}-\frac{23409}{22400}hf_{n+\frac{1}{3}}+\frac{11}{15}hf_{n+\frac{1}{2}}-\frac{17793}{22400}hf_{n+\frac{2}{3}}+\frac{334}{525}hf_{n+\frac{3}{4}}+\frac{10793}{201600}hf_{n+1}
\end{aligned}\right\}\quad(2.7)$$

now, combining (2.5) and (2.7) in the block form to yield the block hybrid method, which can be written explicitly as equation 2.8.

$$
\left.\begin{aligned}
y_{n+\frac{1}{6}} &= y_n + \frac{1}{6}hy'_n + \frac{1000061}{176359680}h^2f_n + \frac{1247}{42525}h^2f_{n+\frac{1}{6}} - \frac{150733}{3444525}h^2f_{n+\frac{1}{4}} + \frac{104833}{3628800}h^2f_{n+\frac{1}{3}} - \frac{409}{45927}h^2f_{n+\frac{1}{2}} + \frac{47623}{10886400}h^2f_{n+\frac{2}{3}} - \frac{5989}{3444525}h^2f_{n+\frac{3}{4}} + \frac{30853}{881798400}h^2f_{n+1}\\
y_{n+\frac{1}{4}} &= y_n + \frac{1}{4}hy'_n + \frac{191741}{20643840}h^2f_n + \frac{85887}{1433600}h^2f_{n+\frac{1}{6}} - \frac{379}{4800}h^2f_{n+\frac{1}{4}} + \frac{599157}{11468800}h^2f_{n+\frac{1}{3}} - \frac{13789}{860160}h^2f_{n+\frac{1}{2}} + \frac{90099}{11468800}h^2f_{n+\frac{2}{3}} - \frac{629}{201600}h^2f_{n+\frac{3}{4}} + \frac{719}{11468800}h^2f_{n+1}\\
y_{n+\frac{1}{3}} &= y_n + \frac{1}{3}hy'_n + \frac{14221}{1102248}h^2f_n + \frac{3874}{42525}h^2f_{n+\frac{1}{6}} - \frac{373376}{3444525}h^2f_{n+\frac{1}{4}} + \frac{617}{8100}h^2f_{n+\frac{1}{3}} + \frac{5314}{229635}h^2f_{n+\frac{1}{2}} + \frac{3853}{340200}h^2f_{n+\frac{2}{3}} - \frac{15488}{3444525}h^2f_{n+\frac{3}{4}} + \frac{311}{3444525}h^2f_{n+1}\\
y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}hy'_n + \frac{1621}{80640}h^2f_n + \frac{27}{175}h^2f_{n+\frac{1}{6}} - \frac{89}{525}h^2f_{n+\frac{1}{4}} + \frac{6399}{44800}h^2f_{n+\frac{1}{3}} - \frac{1}{30}h^2f_{n+\frac{1}{2}} + \frac{783}{44800}h^2f_{n+\frac{2}{3}} - \frac{11}{1575}h^2f_{n+\frac{3}{4}} + \frac{19}{134400}h^2f_{n+1}\\
y_{n+\frac{2}{3}} &= y_n + \frac{2}{3}hy'_n + \frac{18812}{688905}h^2f_n + \frac{9248}{42525}h^2f_{n+\frac{1}{6}} - \frac{796672}{3444525}h^2f_{n+\frac{1}{4}} + \frac{3022}{14175}h^2f_{n+\frac{1}{3}} - \frac{5024}{229635}h^2f_{n+\frac{1}{2}} + \frac{166}{6075}h^2f_{n+\frac{2}{3}} - \frac{34816}{3444525}h^2f_{n+\frac{3}{4}} + \frac{682}{3444525}h^2f_{n+1}\\
y_{n+\frac{3}{4}} &= y_n + \frac{3}{4}hy'_n + \frac{14187}{458752}h^2f_n + \frac{356481}{1433600}h^2f_{n+\frac{1}{6}} - \frac{5841}{22400}h^2f_{n+\frac{1}{4}} + \frac{2827791}{11468800}h^2f_{n+\frac{1}{3}} - \frac{3753}{286720}h^2f_{n+\frac{1}{2}} + \frac{461457}{11468800}h^2f_{n+\frac{2}{3}} - \frac{129}{11200}h^2f_{n+\frac{3}{4}} + \frac{2637}{11468800}h^2f_{n+1}\\
y_{n+1} &= y_n + hy'_n + \frac{109}{2520}h^2f_n + \frac{54}{175}h^2f_{n+\frac{1}{6}} - \frac{128}{525}h^2f_{n+\frac{1}{4}} + \frac{81}{350}h^2f_{n+\frac{1}{3}} - \frac{2}{21}h^2f_{n+\frac{1}{2}} + \frac{27}{1400}h^2f_{n+\frac{2}{3}} - \frac{128}{1575}h^2f_{n+\frac{3}{4}} + \frac{1}{300}h^2f_{n+1}\\
y'_{n+\frac{1}{6}} &= y'_n + \frac{426463}{9797760}hf_n + \frac{65}{189}hf_{n+\frac{1}{6}} - \frac{34292}{76545}hf_{n+\frac{1}{4}} + \frac{11633}{40320}hf_{n+\frac{1}{3}} - \frac{2228}{25515}hf_{n+\frac{1}{2}} + \frac{5149}{120960}hf_{n+\frac{2}{3}} - \frac{1292}{76545}hf_{n+\frac{3}{4}} + \frac{3313}{9797760}hf_{n+1}\\
y'_{n+\frac{1}{4}} &= y'_n + \frac{223577}{5160960}hf_n + \frac{26973}{71680}hf_{n+\frac{1}{6}} - \frac{7727}{20160}hf_{n+\frac{1}{4}} + \frac{156249}{573440}hf_{n+\frac{1}{3}} - \frac{18047}{215040}hf_{n+\frac{1}{2}} + \frac{23571}{573440}hf_{n+\frac{2}{3}} - \frac{47}{2880}hf_{n+\frac{3}{4}} + \frac{1691}{5160960}hf_{n+1}\\
y'_{n+\frac{1}{3}} &= y'_n + \frac{26581}{612360}hf_n + \frac{352}{945}hf_{n+\frac{1}{6}} - \frac{25856}{76545}hf_{n+\frac{1}{4}} + \frac{797}{2520}hf_{n+\frac{1}{3}} - \frac{2216}{25515}hf_{n+\frac{1}{2}} + \frac{319}{7560}hf_{n+\frac{2}{3}} - \frac{256}{15309}hf_{n+\frac{3}{4}} + \frac{41}{122472}hf_{n+1}\\
y'_{n+\frac{1}{2}} &= y'_n + \frac{1733}{40320}hf_n + \frac{27}{70}hf_{n+\frac{1}{6}} - \frac{124}{315}hf_{n+\frac{1}{4}} + \frac{405}{896}hf_{n+\frac{1}{3}} - \frac{1}{210}hf_{n+\frac{1}{2}} + \frac{27}{896}hf_{n+\frac{2}{3}} - \frac{4}{315}hf_{n+\frac{3}{4}} + \frac{11}{40320}hf_{n+1}\\
y'_{n+\frac{2}{3}} &= y'_n + \frac{3329}{76545}hf_n + \frac{352}{945}hf_{n+\frac{1}{6}} - \frac{26624}{76545}hf_{n+\frac{1}{4}} + \frac{124}{315}hf_{n+\frac{1}{3}} - \frac{2848}{25515}hf_{n+\frac{1}{2}} + \frac{113}{945}hf_{n+\frac{2}{3}} - \frac{2048}{76545}hf_{n+\frac{3}{4}} + \frac{32}{76545}hf_{n+1}\\
y'_{n+\frac{3}{4}} &= y'_n + \frac{24849}{573440}hf_n + \frac{26973}{71680}hf_{n+\frac{1}{6}} - \frac{807}{2240}hf_{n+\frac{1}{4}} + \frac{234009}{573440}hf_{n+\frac{1}{3}} - \frac{7083}{71680}hf_{n+\frac{1}{2}} + \frac{101331}{573440}hf_{n+\frac{2}{3}} - \frac{3}{448}hf_{n+\frac{3}{4}} + \frac{39}{114688}hf_{n+1}\\
y'_{n+1} &= y'_n + \frac{151}{2520}hf_n + \frac{256}{315}hf_{n+\frac{1}{4}} - \frac{243}{280}hf_{n+\frac{1}{3}} + \frac{104}{105}hf_{n+\frac{1}{2}} - \frac{243}{280}hf_{n+\frac{2}{3}} + \frac{256}{315}hf_{n+\frac{3}{4}} + \frac{151}{2520}hf_{n+1}
\end{aligned}\right\} \qquad (2.8)
$$

3 ANALYSIS OF COMPUTATIONAL BLOCK HYBRID METHOD

In this section, the analysis of the basic properties of the new method is analyzed. These properties are order, error constant, consistency, zero-stability and region of absolute stability.

Let the linear operator defined on the method be $\ell[y(t);h]$, where,

$$\Delta\{y(t):h\} = A^{(0)}Y_m^{(i)} - \sum_{i=0}^{k}\frac{jh^{(i)}}{i!}y_n^{(i)} - h^{(2-1)}\left[d_i f(y_n) + b_i F(Y_m)\right] \qquad (3.1)$$

Expanding Y_m *and* $F(Y_m)$ in Taylor series and comparing the coefficients of h gives

$$\Delta\{y(t):h\} = C_0 y(t) + C_1 y'(t) + \cdots + C_p h^p y^p(t) + C_{p+1}h^{p+1}y^{p+1}(t) + C_{p+2}h^{p+2}y^{p+2}(t) + \cdots \qquad (3.2)$$

Definition 3.1: The linear operator L and the associate block method are said to be of order p if $C_0 = C_1 = \cdots = C_p = C_{p+1} = 0, \quad C_{p+2} \neq 0.$ C_{p+2} is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+2}h^{p+2}y^{p+3}(t) + 0h^{p+3}$

$$L\{y(t):h\} = C_0 y(t) + C_1 y'(t) + \cdots + C_p h^p y^p(t) + C_{p+1}h^{p+1}y^{p+1}(t) + C_{p+2}h^{p+2}y^{p+2}(t) + \cdots \qquad (3.2)$$

$$\begin{bmatrix} y\sum_{j=0}^{\infty}\frac{\left(\frac{1}{6}\right)^j}{j!}-y_n-\frac{1}{6}hy'_n-\frac{1000061}{176359680}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{1247}{42525}\left(\frac{1}{6}\right)-\frac{150733}{3444525}\left(\frac{1}{4}\right)+\frac{104833}{3628800}\left(\frac{1}{3}\right)-\frac{409}{45927}\left(\frac{1}{2}\right)+\frac{47623}{10886400}\left(\frac{2}{3}\right)-\frac{5989}{3444525}\left(\frac{3}{4}\right)+\frac{30853}{881798400}(1)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{1}{4}\right)^j}{j!}-y_n-\frac{1}{4}hy'_n-\frac{191741}{20643840}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{85887}{1433600}\left(\frac{1}{6}\right)-\frac{379}{4800}\left(\frac{1}{4}\right)+\frac{599157}{11468800}\left(\frac{1}{3}\right)-\frac{13789}{860160}\left(\frac{1}{2}\right)+\frac{90099}{11468800}\left(\frac{2}{3}\right)-\frac{629}{201600}\left(\frac{3}{4}\right)+\frac{719}{11468800}(1)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{1}{3}\right)^j}{j!}-y_n-\frac{1}{3}hy'_n-\frac{14221}{1102248}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{3874}{42525}\left(\frac{1}{6}\right)-\frac{373376}{3444525}\left(\frac{1}{4}\right)+\frac{617}{8100}\left(\frac{1}{3}\right)=\frac{5314}{229635}\left(\frac{1}{2}\right)+\frac{3853}{340200}\left(\frac{2}{3}\right)-\frac{15488}{3444525}\left(\frac{3}{4}\right)+\frac{311}{3444525}(1)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^j}{j!}-y_n-\frac{1}{2}hy'_n-\frac{1621}{80640}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{27}{175}\left(\frac{1}{6}\right)-\frac{89}{525}\left(\frac{1}{4}\right)+\frac{6399}{44800}\left(\frac{1}{3}\right)-\frac{1}{30}\left(\frac{1}{2}\right)+\frac{783}{44800}\left(\frac{2}{3}\right)-\frac{11}{1575}\left(\frac{3}{4}\right)+\frac{19}{134400}h^2 f_{n+1}\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{2}{3}\right)^j}{j!}-y_n-\frac{2}{3}hy'_n-\frac{18812}{688905}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{9248}{42525}\left(\frac{1}{6}\right)-\frac{796672}{3444525}\left(\frac{1}{4}\right)+\frac{3022}{14175}\left(\frac{1}{3}\right)-\frac{5024}{229635}\left(\frac{1}{2}\right)+\frac{166}{6075}\left(\frac{2}{3}\right)-\frac{34816}{3444525}\left(\frac{3}{4}\right)+\frac{682}{3444525}(1)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{3}{4}\right)^j}{j!}-y_n-\frac{3}{4}hy'_n-\frac{14187}{458752}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{356481}{1433600}\left(\frac{1}{6}\right)-\frac{5841}{22400}\left(\frac{1}{4}\right)+\frac{2827791}{11468800}\left(\frac{1}{3}\right)-\frac{3753}{286720}\left(\frac{1}{2}\right)+\frac{461457}{11468800}\left(\frac{2}{3}\right)-\frac{129}{11200}\left(\frac{3}{4}\right)+\frac{2637}{11468800}(1)\right] \\ \sum_{j=0}^{\infty}\frac{(1)^j}{j!}-y_n-hy'_n-\frac{109}{2520}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{54}{175}\left(\frac{1}{6}\right)-\frac{128}{525}\left(\frac{1}{4}\right)+\frac{81}{350}\left(\frac{1}{3}\right)-\frac{2}{21}\left(\frac{1}{2}\right)+\frac{27}{1400}\left(\frac{2}{3}\right)-\frac{128}{1575}\left(\frac{3}{4}\right)+\frac{1}{300}(1)\right] \end{bmatrix}=0$$

Comparing the coefficient of h, according to Skwame et al. (2019), the new method is of uniform order $p=\begin{bmatrix}7 & 7 & 7 & 7 & 7 & 7 & 7\end{bmatrix}^T$ with its error constant are given respectively by $C_{p+2}=\begin{bmatrix}-1.3640\times10^{-10} & -1.3281\times10^{-10} & -1.3497\times10^{-10} & -1.1961\times10^{-10} & -1.4526\times10^{-10} & -1.3561\times10^{-10} & -1.4353\times10^{-09}\end{bmatrix}$

3.2 Consistency of the Method

A numerical method is said to be consistent if the following conditions are satisfied.

i. The order of the method must be greater than or equal to zero to one i.e. $p \geq 1$.

ii. $\sum_{j=0}^{k}\alpha_j = 0$

iii. $\rho(r)=\rho'(r)=0$

iv. $\rho'''(r)=3!\sigma(r)$

Where $\rho(r)$ *and* $\sigma(r)$ are first and second characteristics polynomials of our method. According to Skwame et al. (2019), the first condition is a sufficient condition for the associated block method to be consistent. Hence the new method is consistent.

3.3 Zero Stability of the Method

Definition 3.2: the numerical method is said to be zero-stable, if the roots $z_s,\ s=1,2,\cdots,k$ of the first characteristics polynomial $\rho(z)$ defined by $\rho(z)=\det\left(zA^{(0)}-E\right)$ satisfies $|z_s|\leq 1$ and every root satisfies $|z_s|=1$ have multiplicity not exceeding the order of the differential equation, Skwame et al., (2020). The first characteristic polynomial is given by,

$$\rho(z)=\left|z\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\end{bmatrix}\right|=\left|\begin{bmatrix}z&0&0&0&0&0&-1\\0&z&0&0&0&0&-1\\0&0&z&0&0&0&-1\\0&0&0&z&0&0&-1\\0&0&0&0&z&0&-1\\0&0&0&0&0&z&-1\\0&0&0&0&0&0&z-1\end{bmatrix}\right|=z^6(z-1)$$

Thus, solving for z in

$$z^6(z-1) \tag{3.3}$$

gives $z=0,0,0,0,0,0,1$. Hence the new method is said to be zero-stable.

3.4 Convergence of the Block Method

Theorem 3.1: the necessary and sufficient conditions for linear multistep method to be convergent are that it must be consistent and zero-stable. Hence the new method formulated is consistent Skwame et al. (2019).

3.5 Region of Absolute Stability of our Method

Definition 3.3: the region of absolute stability is the region of the complex z plane, where $z=\lambda h$ for which the method is absolute stable. To determine the region of absolute stability of the block method, the methods that compare neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted. Thus, the method according to Sunday (2018) is called the boundary locus method. Applying this method we obtain the stability polynomial as

$$\bar{h}(w)=\left(-\frac{1}{11612160}w^6-\frac{163}{14631321600}w^7\right)h^{14}+\left(-\frac{810757}{87787929600}w^6+\frac{113}{34836480}w^7\right)h^{12}+\left(-\frac{4473503}{37623398400}w^6-\frac{61}{829440}w^7\right)h^{10} \tag{3.4}$$
$$+\left(-\frac{1235057}{522547200}w^6+\frac{481}{414720}w^7\right)h^8+\left(-\frac{128809}{6967296}w^6-\frac{67}{5184}w^7\right)h^6+\left(-\frac{817}{5040}w^6+\frac{113}{1152}w^7\right)h^4+\left(-\frac{17}{30}w^6-\frac{11}{24}w^7\right)h^2-2w^6+w^7$$

Using the stability polynomial (3.4), we obtain the region of absolute stability in figure below as

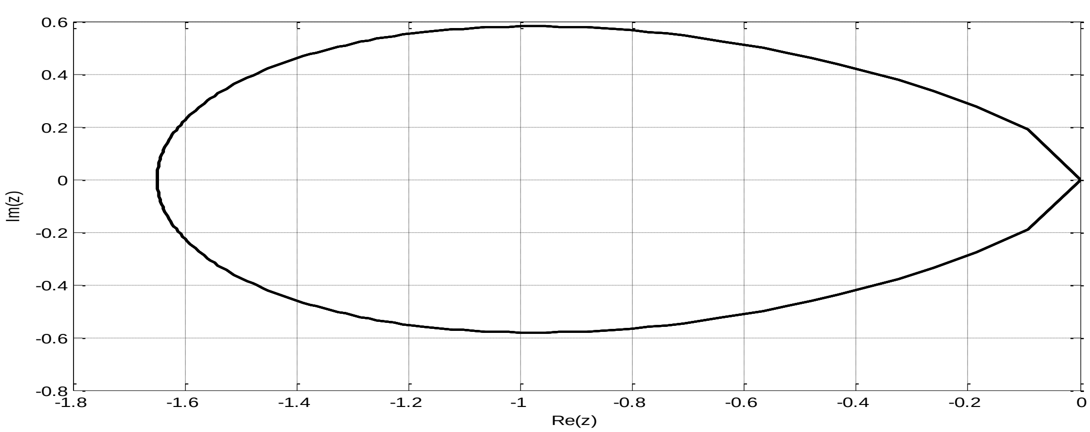

Figure 3.1: Stability region.

4 NUMERICAL APPLICATIONS AND RESULTS

In this section, the effectiveness and accuracy of the proposed method are evaluated by applying it to real-life cases as well as second-order highly stiff initial value problems of the form (1.1), using three test examples. The obtained results are then compared with those of existing approaches, particularly the methods developed by Skwame et al. (2017), Sabo et al. (2021a), Areo and Rufai (2016), Olabode and Momoh (2016), and Lydia et al. (2021).

The tables and figures presented below make use of the following notations: ttt represents the points of evaluation, while ENM denotes the error in the new method. ESBS17 refers to the method proposed by Skwame et al. (2017), and ESKV21a corresponds to the approach developed by Sabo et al. (2021a). Similarly, EAR16 stands for the method of Areo and Rufai (2016), ELJNJ21 represents the contribution of Lydia et al. (2021), and EOM16 refers to the work of Olabode and Momoh (2016).

Problem 4.1:

The problem 4.1 involves a mass-spring system where a weight displaced from equilibrium and subjected to an external force is modelled mathematically, assuming no air resistance, to compute its motion over time, we consider

$$y''(t)=2\sin 4t-16y,\ y(0)=-\frac{1}{2},\ y'(0)=0 \tag{4.1}$$

with the exact solution is given by

$$y(t)=-\frac{1}{2}\cos 4t+\frac{1}{16}\sin 4t-\frac{1}{4}t\cos 4t \tag{4.2}$$

See: Skwame et al., (2017) Sabo, et al., (2021a).

Problem 4.2:

The problem 4.2 models simple harmonic motion of an object stretching a spring, deriving the equation of motion from Newton's law and solving it using the given initial displacement of 18 inches above equilibrium and a downward initial velocity to determine the object's displacement over time. We consider

$$y''(t)=-64y(t),\ y(0)=\frac{3}{2},\ y'(0)=-3 \tag{4.3}$$

We obtain the exact solution as

$$y(t) = -\frac{3}{8}\sin(8t) + \frac{3}{2}\cos(8t) \tag{4.4}$$

See: Areo and Rufai (2016).

Problem 4.3:

The problem 4.3 considers a second-order oscillatory system modelled by the Stiefel linear oscillatory differential equation, where the governing equation of motion is formulated and solved to analyze the displacement of the system over time under specified initial conditions. The equation is

$$y''(t) = 0.001\sin(t) - y'(t),\ y(0) = 0,\ \frac{dy}{dt} = 0.9995 \tag{4.5}$$

With exact solution as

$$y(t) = \sin(t) - 0.0005t\cos(t) \tag{4.6}$$

See: Olabode and Momoh, (2016) and Lydia, et al., (2021).

Table 4.1: Showing the Numerical Results for Problem 4.1

t	**ENM**	**ESBS17**	**ESKV21a**
0.01	0.0000E00	1.6621E-09	1.0000E-19
0.02	3.0000E-20	1.1586E-08	4.1000E-19
0.03	1.0000E-20	2.9743E-08	9.1000E-19
0.04	1.0000E-20	5.6076E-08	1.6600E-18
0.05	1.0000E-20	9.0504E-08	2.6200E-18
0.06	0.0000E00	1.3291E-07	3.8000E-18
0.07	1.0000E-20	1.8317E-07	5.2000E-18
0.08	1.0000E-20	2.4110E-07	6.8500E-18
0.09	1.0000E-20	3.0653E-07	8.7500E-18
0.1	1.0000E-20	1.6621E-09	1.0850E-17

Table 4.2: Showing the Numerical Results for Problem 4.2

t	**ENM**	**EAR16**
0.1	3.0763E-11	3.3496E-07
0.2	1.9247E-10	1.6371E-06
0.3	2.3639E-10	3.2716E-06
0.4	1.4144E-11	3.5979E-06

0.5	4.6555E-10	1.3589E-06
0.6	7.7534E-10	2.9143E-06
0.7	6.0155E-10	6.7226E-06
0.8	9.6414E-11	7.0589E-06
0.9	9.4457E-10	2.6543E-06
1.0	1.3512E-10	4.6056E-06

Table 4.3: Showing the Numerical Results for Problem 4.3

t	**ENM**	**ELJNJ21**	**EOM16**
0.1	3.0000E-20	2.8269E-12	1.0169E-11
0.2	1.9000E-19	5.8994E-12	2.0390E-11
0.3	5.0000E-19	6.8309E-12	1.5451E-13
0.4	9.4000E-19	1.4991E-12	8.1063E-11
0.5	1.5000E-18	1.8395E-12	2.5377E-10
0.6	2.1600E-18	1.6559E-11	5.4848E-10
0.7	2.9200E-18	1.2970E-11	9.9571E-10
0.8	3.7700E-18	8.4312E-11	1.6260E-10
0.9	4.6800E-18	5.3240E-11	2.4697E-10
1.0	5.6400E-18	3.2126E-11	3.5575E-10

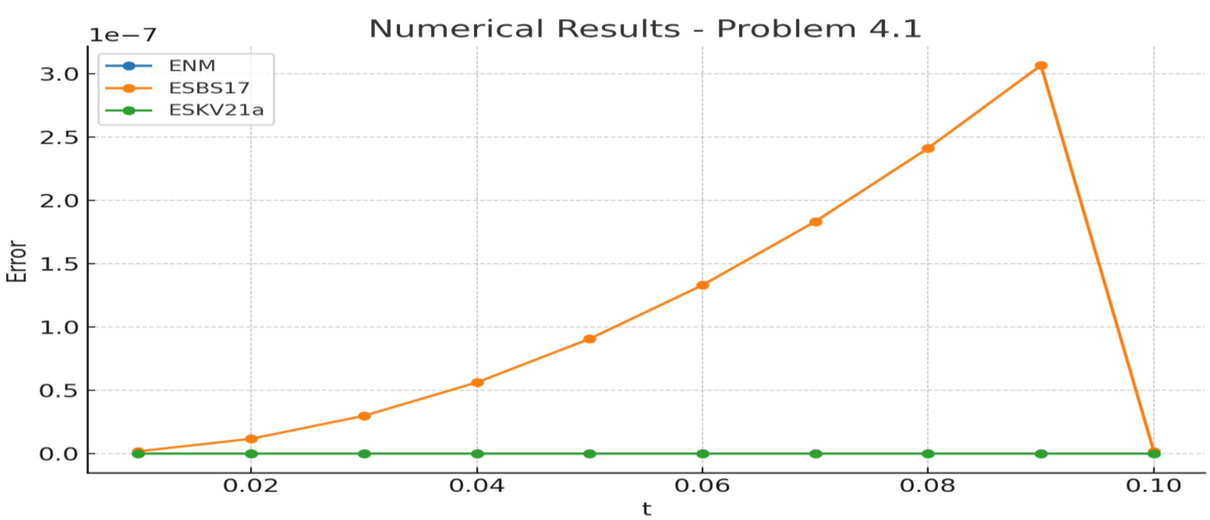

Figure 4.1: Graphical Curve of table 4.1

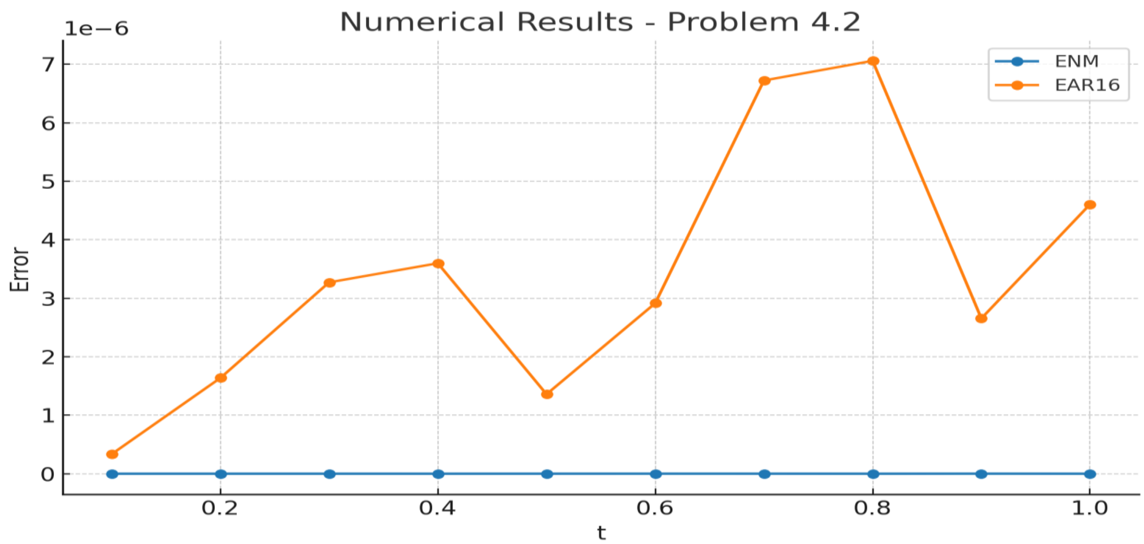

Figure 4.2: Graphical Curve of table 4.2

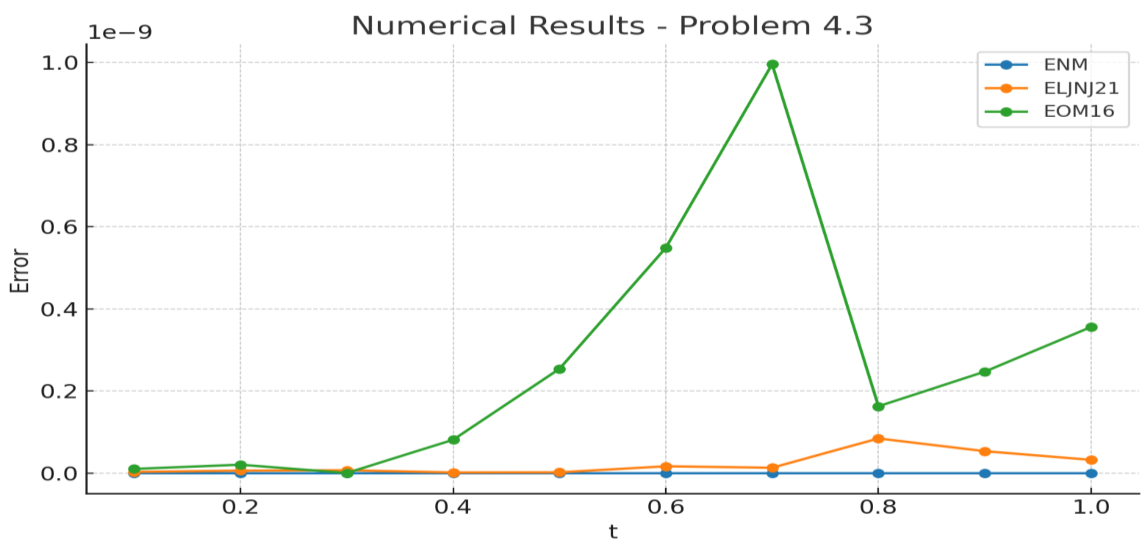

Figure 4.3: Graphical Curve of table 4.3**.**

5. DISCUSSION OF RESULTS

The efficiency and accuracy of the newly proposed method for solving second-order initial value problems (IVPs) were tested using three benchmark problems that represent real-life applications and mathematical models. The performance of the method was evaluated by comparing its results with those of established approaches, including those by Skwame et al. (2017), Sabo et al. (2021a), Areo and Rufai (2016), Olabode and Momoh (2016), and Lydia et al. (2021). These comparisons demonstrate the advantages of the new method in addressing both stiff and oscillatory systems more effectively than the existing alternatives.

Problem 4.1, which models a classical mass-spring system without air resistance, served as a test case for oscillatory motion. The numerical results presented in Table 4.1 and Figure 4.1 show that the new method achieves smaller error magnitudes than the methods of Skwame et al. (2017) and Sabo et al. (2021a). The consistently low error values confirm the method's accuracy and stability in capturing the behavior of oscillatory systems, making it a reliable alternative for dynamical system simulations in physics and engineering.

Problem 4.2 focused on simple harmonic motion derived from Newton's law of motion, with initial displacement and velocity conditions. Results shown in Table 4.2 and Figure 4.2 indicate that the proposed method consistently outperforms the approach of Areo and Rufai (2016), producing smaller numerical errors across the evaluation points. The results highlight the method's strength in solving oscillatory systems while ensuring computational efficiency, reinforcing its adaptability to classical mechanical problems.

Problem 4.3 addressed a more challenging case: a highly stiff oscillatory system modeled by the Stiefel linear oscillatory differential equation. Stiff systems often pose difficulties for numerical methods due to stability concerns. However, as shown in Table 4.3 and Figure 4.3, the proposed method outperformed the techniques of Lydia et al. (2021) and Olabode and Momoh (2016) by producing significantly smaller errors, especially at larger time steps. Taken together, the results from all three problems confirm the robustness, accuracy, and versatility of the new method, underscoring its potential as a reliable tool for solving second-order IVPs across a range of scientific and engineering applications.

6. SUMMARY AND CONCLUSION

This research developed and analyzed a new block hybrid method for the direct solution of second-order initial value problems of ordinary differential equations. The method was designed with improved accuracy and stability, making it suitable for stiff and oscillatory problems that are common in applied sciences and engineering. The theoretical properties of the scheme, including its consistency, zero-stability, and convergence, were established to ensure mathematical validity. The method was then tested on both real-life applications and highly stiff benchmark problems, with its performance compared against existing methods by Skwame et al. (2017), Sabo et al. (2021a), Areo and Rufai (2016), Olabode and Momoh (2016), and Lydia et al. (2021). Numerical results showed that the proposed method consistently produced smaller errors and superior stability behavior compared to the referenced methods, confirming its reliability and effectiveness.

The study concludes that the newly developed block hybrid method is a powerful and efficient numerical tool for solving second-order initial value problems. Its demonstrated accuracy, robustness, and stability, especially in handling stiff systems, make it a significant improvement over several existing approaches. By achieving lower error magnitudes and enhanced convergence, the method not only addresses the limitations of traditional schemes but also provides a practical framework for applications in science and engineering where precise solutions of second-order ODEs are essential. Future work may focus on extending the method to higher-order systems, nonlinear models, and partial differential equations to broaden its applicability.

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